On Joint Transfer of Energy and Information: A Markov Decision Problem Formulation

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***Abstract*— In this Juxtaposition report, we revisit the original paper by Popovski et al. [1] and formulate the same problem as a Markov decision process (MDP). The solution approach is inspired by the classic river crossing problem [2] and its connection to the Bellman equation [3]. Due to the information structure of the problem, a stochastic decision making scenario is presented and the Bellman equation is presented.**

I. Introduction

Optimal control theory serves as a powerful tool for analyzing and interpreting a variety of problems in other fields e.g.: machine learning, reinforcement learning, filtering, biomechanics [4], [5], [6], [7]. In information theory, stochastic control has been used to prove a coding theorem [8] by optimizing directed mutual information and solve more general frameworks as shown by Tatikonda and Mitter [9]. The duality between stochastic optimal control and feedback capacity is stated in [10], [11].

In this report we review the paper by P. Popovski et al. [1] from an optimal control point of view. Intuitively, one may assert that delivering information under energy constraints have some analogies with the classical river crossing problem, whose detailed solution and connection to the dynamic programming are well known [2], [3]. Instead of people crossing a river, data bits flow through a channel throughout some discrete time-horizon subject to given energy constraints. However due to the information structure of the problem—where the message of a node is unknown to the other node—classical dynamic programming is insufficient in finding the optimal policy. Rather, the problem is viewed as a stochastic optimization problem and the Bellman equation is constructed which yields the optimal transmission policy as a solution.

The MDP formulation not only provides a new insight to the problem, but can be used to derive optimal transmission schemes when each node can transmit more than one bit per channel usage. Two immediate useful applications are (1) when nodes are using some quantum channel where qubits, instead of bits, are exchanged, or (2) when each node can send several different quantized energy levels in addition to the information bit. The latter case can be thought of as transmitting some q-ary letter $\{0,\dots,q-1\}$ for each channel usage with each letter in $\{1,\dots,q-1\}$ representing the information bit 1 while conveying distinct energy levels.

The rest of the paper is organized as following: In Sec. II, we present the problem of jointly transferring energy and information and summarize the main result of [1] for the noiseless and lossless case. We also review the river crossing problem and its solutions by following [3], [2] but in modern viewpoint. In Sec. III, we formulate the MDP.

Notations: We denote the set of integers $[m] := \{1,\dots,m\}$ for any positive integer m, and $[0,m] := \{0,\dots,m\}$. The set of binary bit is denoted as $\mathcal{X} = \{0,1\}$. The sequence of variable is succinctly denoted by a superscript: $X^n = (X_1,\dots,X_n)$.

II. Background

A. Interactive Joint Transfer of Energy and Information

We follow [1, Sec. 2]. Consider noiseless-binary communication model between 2 nodes: At each time step denoted by $t \in [n]$, node 1 and 2 can have discrete energy level $(U_{1,t}, U_{2,t}) \in \mathbb{N}^2$ and transmit single bit to each other. If one node has nonzero energy, it can transmit either 0 or 1, while the "1" bit transfers unit energy to the other node. If a node is out of energy, it can only transmit "0" bit. The channel and energy transfer are noiseless and lossless, respectively. (n, R_1, R_2, U) code is defined by following:

1) $U_{1,t} + U_{2,t} = U$ for all $t \in [n]$, i.e. the total energy is conserved;
2) Node 1 has a message $M_1 \in [2^{nR_1}]$ and the source distribution is assumed to be uniform;
3) Node 1 has a encoding function $f_{1,t} : [2^{nR_1}] \times \mathcal{X}^{t-1} \to \mathcal{X}$ that maps the message M_1 and received sequence of bits from node 2 denoted by $\{Y_1,\dots,Y_{t-1}\}$ to the next transmitted bit X_t;
4) Node 1 has a decoding function g_1 that maps its transmitting message M_1 and history of received sequence $\{Y_1,\dots,Y_n\}$ to the estimated message $\hat{M}_2 \in [2^{nR_2}]$;
5) Node 2 also has message, encoder and decoder with similar manner.

Under these assumption, we also have a system dynamics for node 1 by:

$$U_{1,t+1} = (U_{1,t} - X_t) + Y_t, \quad \forall t \in [n-1] \tag{1}$$

and $U_{2,t} = U - U_{1,t}$. A rate pair (R_1, R_2) are jointly achievable for given fixed energy capability U if there exists an (n, R_1, R_2, U) code that achieves asymptotically 0 error probability for sufficiently large n.

B. Main results from Popovski [1]

Special case: If $U = 1$, the optimal strategy achieves sum rate $R_1 + R_2 = 1$ by time-sharing scheme. Each node encodes its message into binary sequences with equal portion of "1" and "0". When one node has energy, then it transmits the codewords through the channel until codeword "1" is

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transmitted and energy is delivered to the other node. Then the channel is always active until both node finish their transmission and achieves sum-rate 1.

Achievability: The Achievable region is given by the following proposition [1, Prop. 1].

Proposition 1 (Inner bound): The rate pair (R_1, R_2) satisfying

$$R_1 \le \sum_{u=1}^{U} \pi_u h_2(p_{1|u})$$
$$R_2 \le \sum_{u=1}^{U} \pi_u h_2(p_{2|u})$$

where $h_2(\cdot)$ is the binary entropy function and $p_{j|u}$ is some parameters take values in $(0,1)$, for $j = 1,2$, $u \in [0,U]$ with $p_{1|0} = p_{2|U} = 0$. π_u is the solution to the following recursion:

$$\pi_u = \pi_u(\phi^u_{1,1} + \phi^u_{2,2}) + \pi_{u-1}\phi^u_{1,2} + \pi_{u+1}\phi^u_{2,1} \tag{2}$$

with $\sum_{u=1}^{U} \pi_u = 1$. $\pi_{-1} = \pi_{U+1} = 0$ by definition. The coefficients $\phi^u_{i,j}$ is the (i,j) element of the following matrix:

$$\phi^u := \begin{pmatrix} (1-p_{1|u})(1-p_{2|u}) & (1-p_{1|u})p_{2|u} \\ p_{1|u}(1-p_{2|u}) & p_{1|u}p_{2|u} \end{pmatrix}$$

The basic idea of proof utilizes "rate-splitting" by constructing U distinct codebooks for each energy level u and transmitting all of them. The overall rate is determined by the size of product space of U codebooks. The proof sketch is summarized in Appendix A.

Converse: The converse part seeks the necessary condition of capacity region for given U. The following proposition [1, Prop. 2], of which proof appear in the Appendix B, illustrates the result:

Proposition 2 (Outer bound): If the rate pair (R_1, R_2) is jointly achievable, then there exists a U-dimensional probability simplex π_u and a joint probability distribution on $\mathcal{X}^2$ for each $u \in [0,U]$ that satisfies $\phi^0_{1,0} = \phi^0_{1,1} = 0$ and $\phi^U_{0,1} = \phi^U_{1,1} = 0$, and the recursion (2) and following set of inequalities are satisfied:

$$R_1 \le \sum_{u=0}^{U} \pi_u H(X_{1|u}|X_{2|u})$$
$$R_2 \le \sum_{u=0}^{U} \pi_u H(X_{2|u}|X_{1|u})$$
$$R_1 + R_2 \le \sum_{u=0}^{U} \pi_u H(X_{1|u}, X_{2|u})$$

where binary random variables $(X_{1|u}, X_{2|u}) \sim \phi^u$.

Remarks: [1, Fig. 2] presents the numerical result that the achievable rate pair with the tightest choice of $p_{j|u}$ and necessary upper-bound are reasonably close. Also it is noted that the optimized proportion $p_{j|u}$ is ordered in u and satisfies symmetry in j, that is, $p_{1|u} = p_{2|U-u}$. Intuitively, this coding strategy utilizes more fraction of "1" if a certain node has relatively high energy level.

C. Dynamic programming and river-crossing problem

The classic "Cannibals and Missionaries" puzzle is concerned with finding the fastest schedule in which 3 cannibals and 3 missionaries can cross the river using one boat that can carry at most 2 people at a time. During the journey, the number of cannibal must not exceed the number of missionaries in either side of the river. R. Bellman gave a generalized mathematical formulation in [3]. Denote the number of cannibals and missionaries on one side of the river as m_1 and n_1 respectively. Likewise, m_2 cannibals and n_2 missionaries are on the other side of the river. The safety constraints are denoted by $R_1(m_1,n_1) \ge 0$ for bank 1, $R_2(m_2,n_2) \ge 0$ for bank 2, and $R_3(x,y) \ge 0$ inside the boat. The capability constraint of the boat is restricted by a constant k. It is observed that the original objective—finding the fastest schedule—may not be well posed, because the safe-crossing may not be possible, and therefore the objective is chosen to seek the maximum number of people that can be transported within a given time horizon.

The optimal control formulation is as follows: Suppose the boat is at bank 1 at the beginning of the stage. The control variable is given by a 4-tuple $u = (x_1, y_1, x_2, y_2)$; at each stage the boat carries x_1 cannibals and y_1 missionaries to bank 2, and then carries x_2 cannibals and y_2 missionaries back from bank 2. A *value function* $f_N(m_1,n_1)$ is defined to be the maximum number of people on bank 2 after N stages. Note that m_2 and n_2 can be obtained as long as the total number of each group is preserved. The Bellman equation is given by

$$f_N(m_1,n_1) = \max_u f_{N-1}(m_1 - x_1 + x_2, n_1 - y_1 + y_2)$$

subject to the following constraints:

$$x_1 \in [m_1], \quad y_1 \in [n_1], \quad 1 \le x_1 + y_1 \le k$$

$$x_2 \in [m_2 + x_1], \quad y_2 \in [n_2 + y_1], \quad 1 \le x_2 + y_2 \le k$$

$$R_1(m_1 - x_1, n_1 - y_1) \ge 0, \quad R_1(m_1 - x_1 + x_2, n_1 - y_1 + y_2) \ge 0$$

$$R_2(m_2 + x_1, n_2 + y_1) \ge 0, \quad R_2(m_2 + x_1 - x_2, n_2 + y_1 - y_2) \ge 0$$

$$R_3(x_1,y_1) \ge 0, \quad R_3(x_2,y_2) \ge 0$$

For $N = 0$, the value function is trivially $f_0(m_1,n_1) = m_2 + n_2$, so we can solve for the optimal control and value function in a recursive manner.

The original safe-crossing objective reduces to finding the minimal N such that

$$f_N(m_1,n_1) = m_1 + n_1 + m_2 + n_2$$

If such N and (m_1,n_1) exist, the optimal safe crossing puzzle can be solved provided that the initial condition is (m_1,n_1).

D. The optimal schedule

In I. Pressman and D. Singmaster's 1989 paper [2], the optimal solution to another version of the river crossing problem, namely "jealous husbands" is presented, as well as the cannibals and missionaries problem. In the jealous husband problem, there are n couples seeking the fastest way to cross a river, and the constraint is that no wife can be with another man without her husbands. In fact, if $n \geq 4$, the constraint cannot be held. The practicable problem assumes the existence of an intermediate island in which the crossing under the constraints becomes possible.

The first main result is that the minimum number of crossings required for the jealous husbands problem with $n \geq 4$ couples and a 2-people boat is given by $8n-6$ if the boat always has to drop an anchor at the island. The proof is simple: the least number of steps to carry $2n$ people is given by $8n-6$ without constraints. Indeed, it can be achieved by De Fontenay's method which consists of 9-step initial move and repeating 8-step intermediate move for $n-3$ times followed by a 9-step final move[1].

Pressman and Singmaster also present that if the direct trip from one bank to the other is permitted, the optimal number of crossing is $4n+1$ if $n>4$. The algorithm consists of 4-step initial move and repeating 4-step intermediate move for $n-2$ times and 5-step final move. Similar to the previous case, the optimality proof employs the fundamental lower bound and shows that it is impossible to improve.

The similar results for cannibals and missionaries problem is also presented: Consider n cannibals and n missionaries crossing the river which has an island, using a boat which can carry 2 people. The optimal number of steps to safely cross the river without bank-to-bank travel is $8n-6$; $4n-1$ if bank-to-bank travel is allowed.

III. MDP Formulation

A. Motivation of considering the dynamic programming

The main contribution of [1] is the suggestion of a communication scheme which achieves the optimal rate given energy constraints. The communication scheme with constraint seems to have an analogy with the river crossing problems: Instead of people crossing the river, data bits are transmitted through the channel, subject to given energy constraint. If the crossing under constraint is possible within $n+o(n)$ steps(cf. II-C) for every combination of codewords, then it might be relevant measure of rate.

However, it turns out that the naive implementation of river crossing idea will not work, because each node has only information of its message and incoming signal up to the previous time step. Therefore, the optimal travel scheduling like II-D is not feasible, but stochastic control setting must be considered.

Recall the rate-splitting and codebook multiplexing strategy: Each node j uses a codebook $C_{j|u}$ $\forall u \in [U]$ comprising $p_{j|u}$ fraction of "1". Given the codebook with optimized $p_{j|u}$ value, approximately the best possible sum-rate in the channel is obtained. In this section we derive that rate-splitting is optimal as a *consequence of an MDP formulation*.

B. Construction of MDP

From heron we focus on node 1 trying to maximize the information being sent over n channel uses. The same analysis holds for node 2 by symmetry. Before communication, the codebooks are generated as in Appendix A and revealed to both nodes. The *state space* is given by possible energy levels $\mathcal{U} = \{0,\dots,U\}$ of node 1. For each state u there is an associated action space $\mathcal{A}_u = \{0,1\}$ $\forall u>0$ and $\mathcal{A}_0 = \{0\}$. Given a codeword C_m in the multiplexed codebook $\mathcal{C}$ the node is to transmit, the goal is to maximize the mutual information between the codeword C_m and the bits transmitted over n channel uses X^n

$$\max_{X^n} I(X^n;C_m). \tag{3}$$

Value function: There is an associated *value function* $J_t^n(u)$ which is the maximum mutual information achieved for the optimal sequence of actions $X_t^{n*} = (X_t^*,\dots X_n^*)$ when the node has energy level u and has been transmitted X^{t-1}:

$$J_t^n(u,x^{t-1}) = \max_{X_t,\dots,X_n} I(X_t,\dots,X_n;C_m|U_t=u,X^{t-1}=x^{t-1}) \tag{4}$$

Remark: Recall that the energy of node 1 follows the difference equation (1):

$$U_{t+1} = (U_t - X_t) + Y_t$$

Thus the information from the incoming signal is captured by the current energy level and the history of actions that has been taken.

The following theorem describes the MDP formulation. The proof appears in the Appendix C

Theorem 1 (Bellman equation): The value function (4) satisfies the Bellman equation

$$\begin{aligned} J_t^n(u,x^{t-1}) = &\max_{x_t\in\mathcal{A}_u} H(Y_t|U_1=u,X^{t-1}=x^{t-1}) \\ &+ \sum_{v=0}^{U} P(U_{t+1}=v|U_t=u,X^t=x^t) J_t^n(v,x^t) \end{aligned}$$

where $x^t = (x^{t-1},x)$, and Y_t is the signal from node 2.

It now remains to determine the transition probability, where the uncertainty lies in the fact that at time t, we are not aware of what the node 2 will send. Node 1 can determine the set of codewords node 2 is sending from at time t by looking at the state-action histories observed by node 1, i.e. given (y_1^t,u_1^t) node 1 can determine $\{c\in\mathcal{C}_2 : y_1^t$ matches $y_1^n(c)$ up to index $t\}$. Notice for each code word $c\in\mathcal{C}_2$, there may be multiple encodings $y_1^n(c)$ due to the fact that node 2 may have sent non-informational 0's because it lacked energy in some instances. If we assume that codewords are uniformly distributed, $P(U_{t+1}|U_t,X_t)$ is given by the fraction of codewords which contain a "1" in the possible codewords.

[1]We do not quote the detailed algorithm for our purpose of presentation. For details, see [2]

C. Solving the MDP

We now have a well-defined finite horizon MDP. This can be solved by value iteration. Here we show some initial conditions that must be satisfied by the value function.

1) $J_t^n(0) = P(Y_t = 0)J_{t+1}^n(0) + P(Y_t = 1)J_{t+1}^n,(1) \quad \forall t \in [n]$
2) $J_n^n(u) = 0, \quad \forall u$
3) The Bellman equation for points other than the initial condition:

$$\begin{aligned} J_t^n(u) =& H(X_t|U_t = u) \\ &+ \max\{J_{t+1}^n(u)P(Y_t = 0) + J_{t+1}^n(u+1)P(Y_t = 1), \\ &J_{t+1}^n(u-1)P(Y_t = 0) + J_{t+1}^n(u)P(Y_t = 1)\} \end{aligned}$$

Once J_t^n is obtained, the optimal transmitting scheme follows.

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Appendix

A. Proof of Prop 1

Let $\mathcal{C}_{j|u}$ be the codebook for each node $j = 1,2$ and each energy level $u \in [U]$. Each codebook $\mathcal{C}_{j|u}$ is made of binary sequences approximately a fraction $p_{1|u}$ of "1" symbols in each codewords. The message is represented by a U-dimensional vector $(m_{j,1},\dots,m_{j,U})$ for each node $j = 1,2$. Suppose each element take value from $m_{j,u} \in [K_{j,u}]$. At each time step, node j send the next codeword corresponding to the current energy level $U_{j,t}$. For node 1, define an ordered set

$$\tau_u = \{t \in [n] \mid U_{1,t} = u\}, \tag{5}$$

τ_u represent the set of time instances when the state takes value at u. For each $u \in [U]$, node 2 takes the first $|\tau_u|$ element from the codeword sequence corresponding to $m_{1,u}$. Node 2 decodes the message by finding $m_{1,u} \in [K_{1,u}]$ that satisfy $\mathcal{C}_{1,u}(m_{1,u})_k = X_{(\tau_u)_k}$ for all $k \in [|\tau_u|]$. Node 2 estimates received signal as $\hat{x}_{1,u} = 1$ if it is not uniquely determined due to $|\tau_u|$ is not large enough.

The error event is union of $\{|\tau_u| < |\mathcal{C}_{1,u}(m_{1,u})|\}$ and two different messages have the same encoding. The proof is achieved by showing that these event has asymptotically zero probability as $n \uparrow \infty$ under given assumption.

B. Proof of Prop 2

$$\begin{aligned} nR_1 &= H(M_1) = H(M_1|M_2, U_{1,1} = u_{1,1}) \\ &= H(M_1, X_1^n, U_1^n|M_2, U_{1,1} = u_{1,1}) \\ &= H(X_1^n, U_1^n|M_2, U_{1,1} = u_{1,1}) \\ &= \sum_{i=1}^n H(X_{1,i}, U_{1,i}|X_1^{i-1}, U_1^{i-1}, M_2, U_{1,1} = u_{1,1}) \\ &= \sum_{i=1}^n H(U_{1,i}|X_1^{i-1}, U_1^{i-1}, M_2, U_{1,1} = u_{1,1}) \\ &+ H(X_{1,i}|X_1^{i-1}, U_1^{i-1}, M_2, U_{1,1} = u_{1,1}) \\ &= \sum_{i=1}^n H(X_{1,i}|X_1^{i-1}, U_1^{i}, M_2, U_{1,1} = u_{1,1}) \\ &\le \sum_{i=1}^n H(X_{1,i}|U_{1,i}, X_{2,i}) \\ &= H(X_1|U_1, X_2, Q) \\ &\le H(X_1|U_1, X_2) \end{aligned}$$

Variable Q uniformly distributed in the set $[1,n]$ and independent of all other variables, along with $X_1 = X_{1Q}$, $X_2 = X_{2Q}$, and $U_1 = U_{1Q}$. Calculating nR_2 is also has same reason of solving nR_1 equation.

$$\begin{aligned} n(R_1 + R_2) &= H(M_1, M_2) \\ &= H(M_1, M_2, X_1^n, X_2^n, U_1^n|U_{1,1} = u_{1,1} \\ &= \sum_{i=1}^n H(U_{1,i}|X_1^{i-1}, X_2^{i-1}, U_1^{i-1}, M_2, U_{1,1} = u_{1,1} \\ &+ H(X_{1,i}, X_{2,i}|X_1^{i-1}, X_2^{i-1}, U_1^{i}, M_2, U_{1,1} = u_{1,1} \\ &\le H(X_1, X_2|U_1) \end{aligned}$$

C. Proof of Theorem 1

Denoting Y_t the bit sent by the opposing node at time t, the mutual information can be written in the form of Dynamic Programming:

$$\begin{aligned} &I(X_t^n; C_m|U_t = u) \\ &= H(X_t^n|U_t = u) + H(C_m|U_t = u) - H(A^n, C_m|U_t = u) \\ &= H(X_t^n|U_t = u) + H(C_m|U_t = u) - H(C_m|U_t = u) \\ &\quad - H(X_t^n|U_t = u, C_m) \end{aligned}$$

$$
\begin{aligned}
&\overset{(a)}{=} H(X_t|U_t=u) - H(a_t|U_t=u,C_m) + H(X_{t+1}^n|U_t=u,X_t) \\
&\quad - H(X_{t+1}^n|U_t=u,C_m,X_t) \\
&= H(X_{t+1}^n, U_{t+1}|U_t=u,X_t) + H(C_m|U_t=u,X_t,U_{t+1}) \\
&\quad - H(X_{t+1}^n, C_m, U_{t+1}|U_t=u,X_t) \\
&= H(X_{t+1}^n|U_t=u,X_t,U_{t+1}) + H(U_{t+1}|U_t=u,X_t) \\
&\quad + H(C_m|U_t=u,X_t,U_{t+1}) - H(X_{t+1}^n, C_m|U_t=u,U_{t+1},X_t) \\
&\quad - H(U_{t+1}|U_t=u,X_t) \\
&= H(X_{t+1}^n|U_{t+1}) + H(Y_t|U_t=u) + H(C_m|U_{t+1},X_t) \\
&\quad - H(X_{t+1}^n, C_m|U_{t+1},X_t) - H(U_{t+1}|U_t=u,X_t) \\
&\overset{(b)}{=} H(Y_t|U_t=u) + H(X_t|U_t=u) \\
&\quad + \sum_v P(U_{t+1}=v|U_t=u,X_t) I(X_{t+1}^n, C_m|U_{t+1}=v,X_t)
\end{aligned}
$$

where (a) the second and third terms of the left hand side cancel and the right hand side follows from using the chain rule for the remaining entropy terms, and (b) $I(X;Y|Z) = H(X|Z) - H(Y|Z) - H(X,Y|Z)$ and $H(X,Y|Z) = H(Y|Z) + H(Y|X,Z)$ gives the result. The last line follows because given a codeword and the energy state, the bit to be sent is deterministic.

In order to maintain the recursive form, take conditioning on X^{t-1} everywhere closes the proof.